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Results from the Holsztynski–Slawny Reduction Method for Ferromagnetic Ising Models

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We establish a variety of results using the Holsztynski–Slawny reduction method to study various ferromagnetic, Ising spin systems. The results range from a new proof of the lack of a first-order phase transition for certain infinite range, pair interaction, one-dimensional systems to a study of certain three-dimensional systems having many-body interactions.

KEY WORDS: Ising spin system; ferromagnetic; phase transition.

1. INTRODUCTION

In a series of papers Slawny[1] and Holsztynski and Slawny[2,3] have established a procedure whereby one ferromagnetic, Ising spin system can be related to another ferromagnetic, Ising spin system. Many times the original system appears to be "reduced" to a simpler, second system; hence, we refer to the method as the HS reduction method. This method has recently been presented by Slawny as part of a review article,[4] where he states that, "although the proofs involve mathematics not familiar to specialists in phase transitions, the final results have a simple enough formulation and the computational criteria are easy to apply." Here we attempt to illustrate the power and ease of the approach by using this single method to obtain a number of results obtained by a variety of methods. In doing so we also extend in some cases these previous results.

The HS reduction method normally involves systems with many-body interactions which have for some time been of interest for many reasons. Interest ranges from obtaining experimental evidence of the presence of such interactions in specific physical systems[5,6] to the mathematical properties of model systems with these interactions present.[7,8]

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Following this introduction, Section 2 contains an introduction to the notation used throughout the paper and a statement of the general HS theorem. This material is in general also contained in the Slawny review article,[4] although one technical change of significance will be noted. Section 3 then follows with a variety of applications of the method and Section 4 contains some concluding remarks.

2. NOTATION AND MAIN THEOREM

We start by defining a general ferromagnetic Ising system. Consider a finite set of sites Λ, which is a subset of the ν-dimensional lattice $\mathbb{Z}^\nu$. On each site in Λ is placed a spin variable σ with $\sigma = \pm 1$ and σ_i representing the spin on the ith site, $i \in \Lambda$. We will consider systems with + boundary conditions, which means all spins outside of Λ have a spin variable $\sigma = +1$. The Hamiltonian of the system is

$$H(\{\sigma\}) = -\sum_{B \in \mathbb{B}} J_B \sigma_B \tag{2.1}$$

where $\mathbb{B}$ is the collection of finite subsets of $\mathbb{Z}^\nu$, $J_B \geqslant 0$, $\{\sigma\}$ is a configuration of the system, and

$$\sigma_B = \prod_{i \in B} \sigma_i \tag{2.2}$$

The thermal average of σ_A is

$$\langle \sigma_A \rangle_\Lambda^+ = Z^{-1} \sum_{\{\sigma\}} \sigma_A e^{-\beta H(\{\sigma\})} \tag{2.3}$$

where Z is the partition function, the superscript on the brackets denotes the + boundary conditions, the subscript the system Λ, and $\beta = 1/kT$. We will be interested in the infinite-volume limit of $\langle \sigma_A \rangle_\Lambda^+$ denoted by $\langle \sigma_A \rangle^+$

To use the HS reduction method, one needs a way to represent subsets of Z. We do this with the "polynomial" notation of Holsztynski and Slawny, where a site $i = (a_1, a_2, \ldots, a_\nu)$ of $\mathbb{Z}^\nu$ is represented by

$$i = X_1^{a_1} X_2^{a_2} \cdots X_\nu^{a_\nu}$$

For one-, two-, and three-dimensional systems we will set $X_1 = X$, $X_2 = Y$, and $X_3 = Z$. Thus, the site $(3, 1, -4)$ will be represented by $X^3 Y / Z^4$. Subsets of sites of $\mathbb{Z}^\nu$ then become polynomials in X, Y, and Z, e.g., the set of sites $(3, 1, -4)$, $(3, 1, 5)$, and $(4, 0, 7)$ is represented by

$$X^3 Y / Z^4 + X^3 Y Z^5 + X^4 Z^7$$

and hence each set $B \in \mathbb{B}$ of (2.1) has a polynomial representation.

We restrict the Hamiltonian to translation-invariant interactions and define a *fundamental* family of bonds to be a set $\mathbb{B}_0 \in \mathbb{B}$ such that any bond $B \in \mathbb{B}$ is a translation of exactly one element of $\mathbb{B}_0$. As an example, for the two-dimensional, nearest neighbor, Ising model on a square lattice in zero magnetic field, $\mathbb{B}_0 = [1 + X, 1 + Y]$ using the polynomial notation.

Next we introduce the idea of the greatest common divisor (g.c.d.) of a set of polynomials and to do so we need to formulate the rules of factoring the polynomial representations. Because the coefficients of the terms making up the polynomials representing collections of sites are all one, multiplication and addition of terms are multiplication and addition in a two-element field, i.e., $0+1=1$, $0+0=1+1=0$, $0 \cdot 1 = 0$, and $1 \cdot 1 = 1$. Thus, for example, $(1+X)^2 = 1 + X^2$ and $1 + X + Y + XY = (1+X)(1+Y)$. For this set of polynomials the g.c.d. is $(1+X)$.

We are now ready to state the major result of the HS reduction method.

Theorem. Let

$$H(\{\sigma\}) = -\sum_{B \in \mathbb{B}} J_B \sigma_B \tag{2.4}$$

be translation-invariant and ferromagnetic. Let $D = \text{g.c.d. } \mathbb{B}$; then the "reduced" Hamiltonian is

$$H'(\{\sigma\}) = -\sum_{B' \in \mathbb{B}'} J_B \sigma_{B'}, \qquad \text{where} \quad \mathbb{B}' = \{B' : B' = B/D,\ B \in \mathbb{B}\} \tag{2.5}$$

and require

$$\sum_{B \in \mathbb{B}_0} n_{B'} J_B < \infty \tag{2.6}$$

where $n_{B'}$ is the number of sites in the set B'. Then

$$P(\beta H) = P(\beta H') \tag{2.7}$$

$$\langle \sigma_{D \cdot A} \rangle^+ = \langle \sigma_A \rangle'^+ \tag{2.8}$$

$$\langle \sigma_A \rangle^+ = 0 \qquad \text{if } A \text{ is not divisible by } D \tag{2.9}$$

Furthermore, at low enough temperatures $\langle \sigma_0 \rangle^+ \neq 0$.

Here $P(\beta H)$ and $P(\beta H')$ are the free energy of the original and the reduced systems, respectively, and the prime on the thermal average in (2.8) means that the average is with respect to the reduced system.

It should be noted that the above theorem differs slightly from the theorems in the Slawny review article.[4] In particular, the review article[4] contains the requirement

$$\sum_{B:\, 0 \in B} J_B < \infty \tag{2.10}$$

while in the above statement we have condition (2.6). Condition (2.6) is more restrictive than (2.10) and the correct statement of the theorem should contain (2.6).[9] For the systems used as examples in the review article,[4] all of which had finite-range interactions, condition (2.6) is not needed; however, its importance when considering infinite-range interaction systems will be illustrated in section 3.2.

3. APPLICATIONS

We now apply the theorem to a variety of ferromagnetic Ising spin systems. In doing so, our emphasis will be on showing the power and computational simplicity of the HS reduction method by reproducing and expanding the results of a number of previous authors who have used a variety of approaches.

3.1. Trivial Systems.

One area where the HS reduction method can be used and where the word reduction is especially appropriate concerns systems recently studied by Mattis and Galler[10] and significantly expanded by Horiguchi and Morita.[11] In the latter reference the authors consider spin systems having "only one kind of spin cluster." By this is meant that the system has multispin interactions, all of which are translations of one basic multispin interaction. No restriction exists on the dimensionality of the system. The above authors directly compute by standard expansion methods the free energy and spin correlation functions for this set of systems.

In the terminology of the HS reduction method these systems are exactly those where the fundamental family of bonds consists of only one element. Such systems have been denoted by Holsztynski and Slawny as trivial systems. The systems can all be reduced to a collection of independent single-spin systems with an external field equal to the interaction strength of the original multispin interaction acting on each spin. For such systems one then immediately has that the free energy per site is $\ln[2\cosh(\beta J)]$, the correlation functions are either zero or products of $\tanh(\beta J)$, and there is no phase transition.

Using a different approach from any of the above papers, Griffiths and Wood[12] pointed out the lack of a phase transition in certain specific systems which fall into this class of systems, e.g., a two-dimensional square lattice with four-body interactions among the spins on the elementary plaquettes. The immediate results mentioned above are contained in Ref. 3. This material, however, was apparently unknown to the authors of the

more recent papers and has been included here to bring together the similar results and to emphasis the directness of the HS reduction method.

In the case of the trivial systems the original system reduces to a collection of independent systems each consisting of a single site. It is easily seen from the HS reduction method that one can construct a class of more complicated systems that still have no phase transition. This is the class of systems that reduce to one-dimensional finite-range interaction systems. All that one needs is that only terms in a single variable X, Y, Z, etc., are not part of the g.c.d. For example, an original system with interactions $(1+X)(1+Y)$ and $(1+X)(1+Y^2)$ reduces to 1 and $(1+Y)$, or in three dimensions $(1+X)(1+Y)(1+Z)$ and $(1+X)(1+Y^2)(1+Z)$ reduce to the same system.

3.2. One-Dimensional Infinite-Range Interaction Systems.

A very different area where the HS reduction method can be applied concerns one-dimensional, infinite-range interaction systems. For finite-range, one-dimensional systems one has no phase transition, but with infinite-range interactions, Dyson[13] has proven the existence of nonzero spontaneous magnetization in certain cases, while Ruelle[14] has proven the absence of a first-order phase transition in certain other cases.

Consider a system with infinite-range pair interactions. We have as the polynomial representation for such interactions $1+X$, $1+X^2$, $1+X^3$,.... Now $1+X^n$ for any integer $n>0$ can be written as $(1+X)(1+X+\cdots+X^{n-1})$, and therefore for a pair interaction system, $(1+X)$ is the g.c.d. Hence, the original system with nearest neighbor, next nearest neighbor,..., etc., pair interaction can be "reduced" to a system with one-body, two-body, three-body,..., interactions as long as condition (2.6) is met. Whether this condition is met depends on the rate of falloff of the interaction strength. For interactions going as J/r^{ε} one has, for (2.6),

$$J \sum_{n=1}^{\infty} n \frac{1}{n^{\varepsilon}} \tag{3.1}$$

and thus (2.5) requires $\varepsilon > 2$. Therefore, for interactions falling off faster than $1/r^2$ the HS theorem of Section 2 can be applied. Since $(1+X)$ is the g.c.d. of the original system, $\langle \sigma_i \rangle^+ = 0$. Furthermore, one has a general result that if the g.c.d. consists of an even number of terms as $(1+X)$ does, then

$$\langle \sigma_A \rangle^+ = 0 \tag{3.2}$$

if $|A|$, the number of sites in A, is odd. This is because no polynomial with an odd number of terms can be factored into two terms either of which has

an even number of terms, due to the characteristics of addition and multiplication in a two-element field. This is similar to a part of the results of Lebowitz,[15] who, using correlation inequalities, looked at the equivalence of different order parameters for Ising ferromagnets and showed that $\langle \sigma_i \rangle^+ = 0$ implies $\langle \sigma_A \rangle^+ = 0$ for $|A|$ odd.

Since Martin-Lof[16] has shown that

$$\langle \sigma_A \rangle^+ = \lim_{h \to 0+} \langle \sigma_A \rangle_h^b \tag{3.3}$$

where b denotes either $+$, $-$, free, or periodic boundary conditions and h is the external magnetic field, we have for $A = i$ that $\langle \sigma_i \rangle^+$ equals the spontaneous magnetization and by the HS reduction method we have shown there is no spontaneous magnetization if the infinite-range pair interactions fall off faster than $1/r^2$. Note that if condition (2.10) needs to be satisfied and not (2.6), then one would have a lack of spontaneous magnetization for all systems with pair interactions falling off faster than $1/r$ rather than $1/r^2$. Yet Dyson[13] has shown that such is not the case, so here we see the significance of the more restrictive (2.6).

3.3. Suzuki's Three-Dimensional Ising Models

Suzuki,[17] using the nonlinear σ–τ transformation

$$\sigma_{i,j,k} = \tau_{i,j,1} \tau_{i,j,2} i,j,k \tag{3.4}$$

studied the three-dimensional Ising model on $\mathbb{Z}^3$ with

$$H_4 = -\sum_{i,j,k} [J\sigma_{i,j,k}\sigma_{i,j,k+1}\sigma_{i+1,j,k}\sigma_{i+1,j,k+1} + J'\sigma_{i,j,k}\sigma_{i,j,k+1}\sigma_{i,j+1,k}\sigma_{i,j+1,k+1}] \tag{3.5}$$

Using the polynomial representation, the fundamental family of bonds can be written as $(1+Z)(1+X)$ and $(1+Z)(1+Y)$. Since $(1+Z)$ is the g.c.d., the model reduces to a collection of independent two-dimensional Ising models with nearest neighbor (hereafter n.n.) pair interactions in the X–Y planes. From the general theorem for the original system (3.5) one has $\langle \sigma_i \rangle^+ = 0$ and thus the spontaneous magnetization is zero for all temperatures. The order parameter for the original system is $\langle \sigma_{i,j,k}\sigma_{i,j,k+1} \rangle^+$ and this is equal to the spontaneous magnetization of the two-dimensional Ising model due to the above reduction. Most of the other results of Suzuki follow from the above reduction and the known results on the two-dimensional n.n. Ising model.

Three particular items are worth noting. First, Suzuki considers adding to (3.5) a pair interaction

$$H_2 = -\sum_{i,j,k} J'' \sigma_{i,j,k} \sigma_{i,j,k+1} \tag{3.6}$$

with $J'' > 0$. Now the fundamental family of bonds has added to it the term $(1+Z)$ and the system reduces to the two-dimensional n.n. Ising model with an external magnetic field equal to J''. No phase transition thus occurs.

Second, Suzuki looks at the two-dimensional system

$$H = -J \sum_{i,j} \sigma_{i,j} \sigma_{i,j+1} \sigma_{i+1,j} \sigma_{i+1,j+1} - J' \sum_{i,j} \sigma_{i,j} \sigma_{i,j+2} \tag{3.7}$$

We simply mention that this system has been used as an example in Slawny's review article[4] and is reducible to the n.n. pair interaction system.

Finally, the third variety of system Suzuki mentions is a 2^n-body interaction system on a $\mathbb{Z}^n$ lattice where the 2^n sites involved in the interaction are those sites on the vertices of the elementary hypercubes of the n-dimensional lattice. Using the σ–τ transformation $n-1$ times, he shows that there is no phase transition. However, this is just the type of system considered in Section 3.1; it consists of only one term in the fundamental family of bonds and hence is what was termed a trivial system. We know immediately that no phase transition occurs and we have the value of all correlation functions.

3.4. The Debierre and Turban Set of Many-Body Interaction Systems

In previous examples we normally began with a system that had been shown to be reducible to a system for which one has a set of known results and these results were then applied to the original system. Here we reverse that situation. Debierre and Turan[18] studied the following set of ferromagnetic many-body interaction systems. Consider a square lattice with Ising spins on each site, $\sigma = \pm 1$, and Hamiltonian

$$H = -J \sum_{i,j} \sigma_{i,j} \sigma_{i,j+1} - J' \sum_{i,j} \prod_{n=0}^{m-1} \sigma_{i+n,j} \tag{3.8}$$

Hence, one has nearest neighbor pair interactions in the vertical direction and m-body interactions in the horizontal direction on the lattice. They studied such systems using a mean-field approach, a phenomenological

renormalization group approach, and a duality approach. All such systems are self-dual and when the phase transition is unique the critical temperature is given by

$$\sinh(2\beta_c J)\sinh(2\beta_c J') = 1 \tag{3.9}$$

so that the critical temperature does not depend on the value of m. Based on their other approaches, the authors conjecture that the order of the phase transition does depend on m and that the $m = 3$ system belongs to the universality class of the $q = 4$ Potts model and gives the border between those systems with first- and second-order phase transitions, i.e., for $m > 3$ one has a transition that is first order at the critical point. Monte Carlo simulations of these systems give results consistent with this conjecture.[19]

The above set of systems are the reduced version of the following set of systems. Consider again a set of Ising spins, $\sigma = \pm 1$, on a square lattice with Hamiltonian

$$H = -J\sum_{i,j}\sigma_{i,j}\sigma_{i+1,j}\sigma_{i,j+1}\sigma_{i+1,j+1} - J'\sum_{i,j}\sigma_{i,j}\sigma_{i+n,j} \tag{3.10}$$

This system has as its fundamental family of bonds $(1+X)(1+Y)$ and $(1+X^n)$. Since, as we saw in Section 3.2, $(1+X^n) = (1+X)(1+X+X^{n-1})$, then we have this set of systems reducing to the previous set of systems. With this set of systems the transitions become first or second order, depending on the range of the pair interaction as opposed to the number of sites involved in the many-body interaction of (3.8). We mention this because it may be easier to either prove or disprove the conjecture of Debierre and Turban.

4. CONCLUSION

At this point we note that obviously one can endlessly construct many-body interaction systems that reduce to previously solved Ising spin systems, e.g., the eight-vertex model, the Baxter–Wu model, etc. We have not done so because many of these initial systems are rather contrived and artifical. Rather than do this, we have tried to direct attention to such systems where both initial and reduced systems have been previously studied by various authors.

It is hoped that this presentation of the HS reduction method, general theorem, and sample of results has shown the power and simplicity of the method and has bridged the gap between some of the previous, less mathematical papers and the more mathematical papers of Slawny and Holsztynski.

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REFERENCES

1. J. Slawny, *Commun. Math. Phys.* **46**:75 (1976).
2. W. Holsztynski and J. Slawny, *Lett. Nuovo Cimento* **13**:534 (1975).
3. W. Holsztynski and J. Slawny, *Commun. Math. Phys.* **66**:147 (1979).
4. J. Slawny, in *Phase Transitions and Critical Phenomena*, Vol. 11, C. Domb and J. L. Lebowitz, eds. (Academic Press, 1986).
5. J. A. Barker, *Phys. Rev. Lett.* **57**:230 (1986).
6. H.-Y. Kim and M. W. Cole, *Phys. Rev. B* **35**:3990 (1987).
7. M. Kolb and K. A. Penson, *J. Phys. A: Math. Gen.* **19**:L779 (1986).
8. J. L. Monroe, *J. Phys. A: Math. Gen.* **19**:2499 (1982).
9. J. Slawny, private communication.
10. D. C. Mattis and R. Galler, *Phys. Rev. B* **27**:2894 (1983).
11. T. Horiguchi and T. Morita, *Phys. Lett.* **105a**:57 (1984).
12. H. P. Griffiths and D. W. Wood, *J. Phys. C: Solid State Phys.* **6**:2533 (1974).
13. F. J. Dyson, *Commun. Math. Phys.* **12**:91 (1969).
14. D. Ruelle, *Commun. Math. Phys.* **9**:267 (1968).
15. J. L. Lebowitz, *J. Stat. Phys.* **16**:3 (1977).
16. A. Martin-Lof, *Commun. Math. Phys.* **24**:253 (1972).
17. M. Suzuki, *Phys. Rev. Lett.* **28**:507 (1972).
18. J. M. Debierre and L. Turban, *J. Phys. A: Math. Gen.* **16**:3571 (1983).
19. H. W. J. Blote, A. Compagner, P. A. M. Cornelissen, A. Hoogtand, F. Mallezie, and C. Vanderzande, *Physica* **139A**:395 (1986).